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Impedance Profile Inversion via the First Transport Equation*

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The subject of this paper is the inverse reflection problem for a stratified elastic half-space. That is, a linear elastic medium, whose elastic properties depend only on depth from a planar free surface, is stimulated at $t = 0$ by a plane wave impulsive source. The motion of a typical surface element is recorded for $0 \leqslant t \leqslant 2T$. It is shown that this *surface trace* determines the *acoustic impedance* of the medium as a function of *travel time*, to (travel-time) depth T. Moreover, we give a precise characterization of those functions which may appear as surface traces, and show uniqueness, existence, and continuous dependence of the logarithm of the impedance as a function of the surface trace in the Sobolev H^1 topology.

1. INTRODUCTION

The subject of this paper is the inverse reflection problem for a stratified elastic half-space. That is, a linear elastic medium, whose elastic properties depend only on depth from a planar free surface, is stimulated at $t = 0$ by a plane wave impulsive source. The motion of a typical surface element is recorded for $0 \leqslant t \leqslant 2T$. We show that this *surface trace* determines *acoustic impedance* of the medium as a function of *travel time*, to (travel-time) depth T. Moreover we give a precise characterization of those functions which may appear as surface traces, and show uniqueness, existence, and continuous dependence of the logarithm of the impedance as a function of the surface trace in the Sobolev H^1 topology.

The proofs are designed to supply the pattern for stability results for numerical algorithms to give approximate solutions to the problem. Numerical experiments with one such class of algorithms are in progress. These numerical stability and experimental results will be reported elsewhere.

The main conceptual device of the present paper is the reduction of the inverse reflection problem to a nonlinear hyperbolic boundary value problem for the displacement field. Other work on this problem, including our own

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previous work, has involved (either explicitly or implicitly) a similar boundary value problem for the velocity field. The present approach offers important simplifications and technical advantages. Local solutions to this boundary value problem are produced by iteration, then extended to global solutions via a priori estimates. These estimates are the most important point of this paper; they show very clearly the connection between the (stable) solvability of the inverse reflection problem and a transmission property of media with parameters of bounded variation, which we call *acoustic transparency*. We give a physical interpretation of this property, and propose a measure for it, which we call the *modulus of* (acoustic) *transparency*. This number figures prominently in the main a priori estimate.

The paper is organized as follows: In Section 2, we state and discuss the main results, including relation to previous work. In Section 3, we use geometric optics to formulate the inverse reflection problem as a boundary value problem. In Section 4, we discuss the notion of acoustic transparency and show that it is related to a property of the impulse–response trace (which is the data of the inverse problem). In Section 5, we combine transparency with a simple energy identity to produce the main a priori estimate and prove the main results of the paper (Theorems 1 and 2) *modulo* the local construction of solutions to the boundary value problem. This local construction is carried out in Section 6, using "sideways" energy estimates and a contraction mapping argument.

2. Statement and Discussion of Results

Consider an isotropic elastic stab of thickness l in the z (vertical) direction, and of infinite horizontal extent. Suppose that the density ρ and the Lame constants λ, μ are functions of the depth coordinate z only. Then the linear elastic wave equations governing small displacements from equilibrium have solutions which are functions of z, t only, which we shall call plane waves. Such a plane wave motion $u(z, t)$, which necessarily has infinite energy but may have finite energy density *per* horizontal surface element, obeys the one-dimensional wave equation

$$(\rho\partial_t^2 - \partial_z E \partial_z)\, u = f, \tag{2.1}$$

where $E = \lambda + 2\mu$ (compressional wave), or $E = \mu$ (shear wave), and f is the impressed body force, which necessarily also must be a plane wave, $f = f(z, t)$.

We shall be concerned in this article with the impulse–response u, which is a solution to (2.1) with $f(z, t) = \delta(z)\,\delta(t)$. Such a source gives rise to a motion with infinite horizontal energy density on the wave front $\{z = t\}$, and can only be approximated in laboratory experiments. However, we do not discuss the consequences of this important observation in this paper.

We suppose that the surface $\{z=0\}$ is free, i.e.,

$$\partial_z u = 0, \qquad z = 0 \tag{2.2}$$

and that the surface $\{z=l\}$ has some appropriate boundary condition, for instance,

$$\partial_z u + \zeta \partial_t u = 0, \qquad z = l. \tag{2.3}$$

The selection $\zeta = -1$ would correspond to impedance matching of the medium in $[0, l]$ with another elasticum in $\{z > l\}$, for example. The precise form of the boundary condition at $\{z = l\}$ turns out to be unimportant for our purposes.

We finally suppose that the system is in its equilibrium state for $t < 0$,

$$u(x, t) \equiv 0, \qquad t < 0. \tag{2.4}$$

System (2.1)–(2.4) is well posed in the sense of distributions, and has a unique distribution solution, which we shall call the *impulse–response displacement field*, when $\rho, E > 0$ are bounded and measureable.

The inverse reflection impulse–response problem (which we shall call the *inverse problem* in the remainder of this paper) is: given the time history of a typical element of the displacement field at the free surface $\{z=0\}$ (the *displacement trace*) infer whatever may be inferred concerning the elastic properties of the interior of the medium, i.e., the functions ρ, E.

To state our result, we introduce the local index of refraction or signal speed

$$c = E^{1/2} \rho^{-1/2}.$$

The acoustic impedance is defined by

$$\eta = E^{1/2} \rho^{1/2} = E c^{-1}.$$

We normalize $\eta(0) = 1$, as we may do by properly choosing the scale in which the displacement u is measured. The travel-time function is given by

$$T(z_1, z_2) = \int_{z_1}^{z_2} c^{-1}.$$

The one-way travel time through the slab is $T_1 = T(0, l)$. Since $c > 0$, we may use the travel time from the surface $\{z = 0\}$ as a new depth variable

$$x = x(z) = \int_0^z c^{-1} = T(0, z).$$

Various authors have proven versions of the following statement (see [1–6]):

> *The trace* $\{u(0,t): 0 < t \leqslant 2T_1\}$ *of the impulse response determines the acoustic impedence* η *as a function of the travel time variable* x, $0 \leqslant x \leqslant T_1$.

In particular, the trace of the impulse response does not determine both ρ and E, but merely the combination $\eta(x)$. (See remarks at the beginning of [5].) Also, since reflections from $\{x = l\}$ do not arrive at $x = 0$ before $t = 2T_1$, the nature of the boundary condition at $\{x = l\}$ clearly does not figure in any such result.

The various versions of this statement appearing in the literature differ in their smoothness requirements on ρ, E, in the completeness with which they characterize the class of traces $\{u(0,t)\}$, and in the constructiveness of the proofs.

We shall prove the following version of the statement:

THEOREM 1. *The function* $g: [0, 2T_1] \to \mathbb{F}$ *is the trace of the impulse–response of a stratified elastic slab in the travel-time range* $\{0 \leqslant z \leqslant T_1\}$ *with* $\eta \in H^1[0, T_1]$ *if and only if:*

(i) $g \in H^1[0, 2T_1]$,

(ii) *extend g to be an odd function on* $[-2T_1, 2T_1]$. *Then*

$$\lim_{t\to 0+} g(t) = \lim_{t \to 0-} g(t) + 2$$

and g is in H^1_{loc} *on any interval not containing* 0.

(iii) *Let G be the Hilbert–Schmidt operator, with distribution kernel* $g'(s-t)$, *on* $L^2[-T_1, T_1]$. *Then there is* $\varepsilon > 0$ *so that*

$$G \geqslant \varepsilon.$$

The number ε appearing in condition (ii) is related to a property we call *acoustic transprency*: if $\eta \in H^1$ (in fact, if $\eta \in BV$, see [9]), then any plane-wave disturbance u of positive energy density

$$\frac{1}{2}\int_0^l dx[\rho(\partial_t u)^2 + E(\partial_x u)^2]|_{t=0}$$

propagated forward and backward in time according to the wave equation, causes a nontrivial disturbance on the surface $\{x = 0\}$ within the time interval $[-T_1, T_1]$.

We remark that this notion is extremely important for understanding inverse reflection and similar problems. The definition of acoustic transparency given here is, moreover, appropriate *only* for one-dimensional problems, and must be modified to apply to several dimensional wave propagation problems. This point is discussed in the forthcoming paper [10].

Proof. The proof of Theorem 1 is constructive. We show first that the impulse–response u, together with the square root of the impedance $\beta = \eta^{1/2}$, satisfy a nonlinear boundary value problem

$$[\partial_t^2 - \partial_x^2 - 2(\partial_x \log \beta)\,\partial_z]\, u = 0 \qquad \text{in} \quad 0 < x < t < 2T_1 - x, \quad (2.5\text{a})$$

$$u(0, t) = g(t), \qquad \partial_x u(0, t) = 0, \qquad 0 < t < 2T_1, \quad (2.5\text{b})$$

$$\lim_{\tau \to 0} u(x, x + \tau) = \beta(x)^{-1}. \quad (2.5\text{c})$$

Equation (2.5c), the *first transport equation* of geometric optics, determines the leading singularity of u. This problem, interpreted as an evolution system in z, has local solutions by an easy contraction mapping argument (Section 6). Once ε has been interpreted as a measure of acoustic transparency, simple energy inequalities imply a priori estimates which allow the extension of local into global solutions, which completes the proof of Theorem 1.

The method of proof also leads to the Lipschitz estimate for the map $g \to \eta$ contained in

THEOREM 2. (i) *Suppose g satisfies conditions* (i)–(iii) *of Theorem* 1. *Then there exists $\delta > 0$ for which all $\tilde{g} \in H^1[0, 2T_1]$ with*

$$\| g - \tilde{g} \|_{H^1} < \delta$$

and $\tilde{g}(0^+) = 1$ satisfy (i)–(iii).

(ii) *Suppose g, $\tilde{g}$ as in part* (i). *Then there exists $c > 0$, depending on ε and on $\| g \|_{H^1}$, so that for the corresponding impedances η, $\tilde{\eta}$,*

$$\| \eta - \tilde{\eta} \|_{H^1} \leqslant C \, \| g - \tilde{g} \|_{H^1},$$

C is a smooth increasing function of ε^{-1}, $\| g \|_{H^1}$.

For impedances of class H^1, the inverse problem is thus well posed in the classical sense.

We close this section with a brief discussion of the relation of the present results to previous work. The bulk of previous results on the one-dimensional elastic inverse problem, exemplified by [2, 3, 6], have hypothesized two derivatives of the impedance. Two (L^2-) derivatives are necessary to ensure the progressing wave expansion of the *velocity* impulse response (see [6, Sect. 2]), or—what amounts to the same thing—to obtain an integral equation in the style of Gel'fand–Levitan–Marchenko, with or without the use of the Liouville transformation. Thus the present results are out of reach of such techniques. Carroll and Santosa [4] consider the displacement field, as we do here, but use global transform techniques and a version of the Gel'fand–Levitan integral equation. They relate g to η when $\eta \in C^1$, but do not give stability estimates. Gerver [1] and Bamberger *et al.* [5] treat the inverse problem for $\eta \in BV$, on the other hand, by regularization techniques from the theory of ill posed problems. In particular, in [5] the problem is treated numerically as a constrained optimization problem.

In our previous paper [6], we gave uniqueness, existence, and continuous dependence results valid when $\eta \in H^2$ (in the paper, $\rho \in C^2$, $E \equiv 1$, but the extension is easy). These depended on the progressing wave decomposition of the velocity field, in particular on the second transport equation, and a version of the Gel'fand–Levitan nonlinear Volterra equation. As in most treatments of inverse problems by way of Gel'fand–Levitan ideas, the solution of the problem was obscured by the machinery. The advantage of the present treatment is thus partly conceptual. Also, the estimates on and construction of the solution are considerably simpler, and valid under the weaker hypothesis $\eta \in H^1$.

The boundary value problem (2.5) also provides a way to treat the inverse problem numerically. The final advantage of the present approach is its provision of a pattern for the proof of stability results for simple numerical schemes for (2.5), allowing the approximate solution of the inverse problem with rigorous error bounds. Our previous paper [8] on numerical stability for a related problem, which rested on the results [7], did not provide a clear prescription for the construction of stable algorithms for the inverse problem. For a discussion of the nontriviality of this point and results of numerical experiments on the present problem, see [11]. We shall show in a future paper that the proofs of Theorems 1 and 2 lead to quite general numerical stability results—as might be expected, since these are essentially energy methods.

The only important unanswered question about the inverse problem, as formulated in this paper, is whether a direct (nonregularization) treatment of the stability question for $\eta \in BV$ is possible. We imagine that a continuous dependence result for BV-layered media would be stated relative to some weak topology.

3. Formulation of Inverse Problem as a Boundary Value Problem

For smooth η, the well-known progressing wave expansion of Lax and Courant shows that the unique distribution solution of

$$\begin{aligned} (\eta\partial_t^2 - \partial_x\eta\partial_x)\, u &= \delta(x)\,\delta(t), \\ u &\equiv 0, \qquad t < 0, \\ \partial_x u &= 0, \qquad x = 0, \end{aligned} \tag{3.1}$$

is smooth inside the light cone $\{(x, t): 0 \leqslant x \leqslant t\}$ and satisfies the boundary condition

$$\lim_{\tau \to 0} u(t, t + \tau) = \eta^{-1/2}(t)$$

(see [12, Chap. VI, Sect. 4] or [6, Sect. 2]). Using straightforward limiting arguments and energy estimates, one can prove

Proposition 1. *Suppose $\eta \in H^1(0, T)$, $\eta > 0$. Then the problem*

$$\begin{aligned} \eta\partial_t^2 u - \partial_x\eta\partial_x u = 0 \qquad &\text{in} \quad \{(x, t): 0 \leqslant x \leqslant t, x \leqslant t \leqslant 2T - x\}, \\ u(t, t) = \eta^{-1/2}(t), \qquad &0 \leqslant t \leqslant T, \\ \partial_x u(0, t) = 0, \qquad &0 \leqslant t \leqslant 2T, \end{aligned} \tag{3.2}$$

has a unique solution satisfying

$$\begin{aligned} &u(\cdot, t) \in H^1[0, \min(t, 2T - t)], \\ &\partial_t u(\cdot, t) \in L^2[0, \min(t, 2T - t)] \qquad \text{for} \quad 0 \leqslant t \leqslant 2T, \end{aligned} \tag{3.3}$$

and

$$\|\partial_x u(\cdot, t)\|_{L^2}^2 + \|\partial_t(u(\cdot, t)\|^2 \leqslant \|\partial_x \eta\|_{L^2}.$$

Moreover, if u is extended to be identically zero outside of the forward light cone $\{0 \leqslant x \leqslant t\}$, then u is the unique distribution solution to problem (3.1).

It follows that the impulse–response trace $g(t) = u(0, t)$, $0 < t \leqslant 2T$, can be calculated by solving problem (3.2) for given $\eta(x)$, $0 \leqslant x \leqslant T$. This approach to calculating g has been implemented numerically—for details, see the forthcoming paper [11].

Note that, if $g = u(0, \cdot)$ is also given, then (3.2) becomes overdetermined. This suggests the following statement of the inverse problem:

Given $g(t)$, $0 \leqslant t \leqslant 2T$, find u and η so that

$$\begin{aligned} \eta \partial_t^2 u - \partial_x \eta \partial_x u = 0 \qquad & \text{in} \quad \{(x, t): 0 \leqslant x \leqslant t \leqslant 2T - x\}, \\ u(0, t) = g(t), & \\ \partial_x u(0, t) = 0, \qquad & 0 < t \leqslant 2T, \\ u(z, z) = \eta^{-1/2}(z). & \end{aligned} \tag{3.4}$$

An immediate consequence of Proposition 1 is

COROLLARY 1. *Suppose (u, η) solve (3.4) with $\eta \in H^1[0, T]$, $\eta > 0$, and u satisfying (3.3). Then $g \in H^1[0, 2T]$ is the trace of the impulse response of the stratified elasticum with impedance η.*

Thus a solution (u, η) of (3.4) amounts to a solution of the inverse problem.

4. ACOUSTIC TRANSPARENCY AS A PROPERTY OF THE IMPULSE–RESPONSE TRACE

Define the *trace operator* S on $\mathscr{H} = H^1[0, T] \oplus L^2[0, T]$:

$$S: \mathscr{H} \to L^2[-T, T] \quad .$$

by $S(u_0, v_0) = \partial_t u(0, \cdot)$, where u solves

$$\begin{aligned} \eta \partial_t^2 u - \partial_x \eta \partial_x u = 0 \qquad & \text{in} \quad \{(x, t): 0 \leqslant x \leqslant T - |t|\}, \\ \partial_x u(0, t) = 0, \qquad & |t| \leqslant T, \end{aligned}$$

and

$$u(x, 0) = u_0(x), \qquad \partial_t v(x, 0) = v_0(x).$$

By slightly modifying the proof of the trace theorem in [9], one can prove

THEOREM 3. *Suppose* $\log \eta \in BV[0, T]$. *Then S is a bounded operator with bounded inverse, and one can estimate $\|S\|$, $\|S^{-1}\|$ in terms of* $\|\log \eta\|_{BV}$.

We provide $\mathscr{H}$ with the "energy" Hermitian form $\langle\ ,\ \rangle_E$, defined by

$$\langle (u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \rangle_E = \int_0^T dx \eta(x) (\partial_x \bar{u}_0(x)\, \partial_x \tilde{u}_0(x) + \bar{v}_0(x)\, \tilde{v}_0(x)).$$

In fact

$$\tfrac{1}{2}\,\|(u_0, v_0)\|_E^2 = \tfrac{1}{2}\langle (u_0, v_0), (u_0, v_0)\rangle_E$$
$$= \tfrac{1}{2}\int_0^l dz\{\rho(z)\,|v_0(z)|^2 + E(z)\,|\partial_x u_0(z)|^2\},$$

which gives the mechanical energy density per unit horizontal area of the horizontally stratified disturbance with displacement u_0 and velocity v_0.

It follows from Theorem 3 that, for any $(u_0, v_0) \in \mathscr{H}$, we have

$$\|S(u_0, v_0)\|^2_{L^2[-T,T]} \geqslant \varepsilon\,\|(u_0, v_0)\|_E^2, \tag{4.1}$$

where

$$\varepsilon = \|S^{-1}\|^{-2},$$

where the norm is the operator norm of S^{-1}, viewed as an operator from $L^2[-T, T]$, with the usual inner product to $(\mathscr{H}, \langle\,,\,\rangle_E)$.

Inequality (4.1) shows that a plane wave initial disturbance with positive energy density in the slab will always give rise to a nontrivial disturbance on the surface $\{z = 0\}$ within the time interval $[-T, T]$. Consequently, we shall call ε the *modulus of (vertical) acoustic transparency* of the slab. We emphasize that ε is bounded below by a positive decreasing function of $\|\log \eta\|_{BV} \leqslant T^{1/2}\,\|\log \eta\|_{H^1}$.

We assume for the moment that η is smooth. Then it is quite standard that the singular initial value problem

$$\eta\partial_t^2 u - \partial_x \eta \partial_x u = 0, \qquad \partial_x u(0, \cdot) = 0,$$
$$u(x, t_0) = 0, \qquad \partial_t u(x, t_0) = \eta^{-1}(x_0)\,\delta(x - x_0)$$

has a unique solution, which we shall denote by $R(x, t; x_0, t_0)$. This distribution, called the Riemann function, has the following properties:

PROPOSITION 2.

$$R(x, t; x_0, t_0) = R(x, t - s; x_0, t_0 - s). \tag{4.2}$$

R is odd in $t - t_0$. (4.3)

$R(x, t; 0, 0)$ is identical to the impulse-response for $t > 0$. (4.4)

If u is a smooth solution of the wave equation, then

$$u(x, t) = \int dx_0\,\eta(x_0)\{D_2 R(s, t; x_0, t_0)\,u(x_0, t_0)$$
$$+ R(x, t; x_0, t_0)\,D_2 u(x_0, t_0)\}. \tag{4.5}$$

$R(x, t; \cdot, \cdot)$ solves

$$(\eta(x_0)\,\partial_{t_0}^2 - \partial_{x_0}\eta(x_0)\,\partial_{x_0})\,R(x, t; x_0, t_0) = 0$$
$$D_3 R(x, t; 0_0, t_0) = 0$$
$$R(x, t_0; x_0, t_0) = 0$$
$$D_4 R(x, t_0; x_0, t_0) = -\eta(x_0)\,\delta(x - x_0). \tag{4.6}$$

$$R(x, t; x_0, t_0) = -R(x_0, t_0; x, t) = R(x_0, t; x, t_0). \tag{4.7}$$

$$S(u_0, v_0)(t) = \partial_t \int dx_0\,\eta(x_0)\{\partial_t R(0, t; x_0, 0)\,u_0(x_0) + R(0, t; x_0, 0)\,v_0(x_0)\}. \tag{4.8}$$

$$\begin{aligned} R(x, s+t; x_0, 0) &= \int dy_0\,\eta(y_0)\{D_2 R(x, s+t, y, t)\,R(y, t; x_0, 0) + R(x, s+t, y, t)\,D_2 R(y, t; x_0, 0) \\ &= \int dy\,\eta(y)\{D_2 R(x, s; y, 0)\,R(y, t; x_0, 0) + R(x, s; y, 0)\,D_2 R(y, t; x_0, 0)\}. \end{aligned} \tag{4.9}$$

Remark. Identity (4.9) expresses what Hadamard [13] called "Huyghens' major premise," which amounts to the statement that the solution operators for the wave equation form a group. Formally, at least, this is true because the coefficients of the wave operator are independent of t. We have pointed out in [7] how (4.9) is related to the nonlinear Volterra equation of Gel'fand and Levitan [14].

LEMMA 1. *The adjoint of $S: (\mathscr{H}, \langle\,,\,\rangle_F) \to L^2[-T, T]$ is given by*

$$S^{\dagger}\phi = \left(\int_{-T}^{T} dt R(\cdot, t; 0, 0)\,\phi(t),\ -\int_{-T}^{T} dt\, D_2 R(\cdot, t; 0, 0)\,\phi(t)\right).$$

Proof. According to the progressing wave expansion of the impluse-response (3.2), and (4.4), $R(\cdot, \cdot; 0, 0)$ is square integrable in $[0, T] \times [-T, T]$, whereas the second integral has the form of a convolution operator plus a Hilbert–Schmidt operator, as does the x derivative of the first integral. It remains to veiify the adjoint property:

$$\langle \phi, S(u_0, v_0)\rangle_{L^2[-T,T]}$$
$$= \int_{-T}^{T} dt\bar{\phi}(t)\,\partial_t \int_0^T dy\,\eta(y)\{D_2 R(0, t; y, 0)\,u_0(y)$$

$$
\begin{aligned}
&+ R(0, t; y, 0)\, v_0(y)\}, \\
&= \int_{-T}^{T} dt\bar{\phi}(t) \int_0^T dy\eta(y)\{\partial_t^2 R(0, t; y, 0)\, u_0(y) \\
&\quad + R(0, t; y, 0)\, v_0(y)\}, \\
&= \int_{-T}^{T} dt\bar{\phi}(t) \int_0^T dy\{\partial_y \eta(y)\, \partial_y R(0, t; y, 0)\, u_0(y) \\
&\quad + \eta(y)\, \partial_t R(0, t; y, 0)\, v_0(y)\}, \\
&= \int_{-T}^{T} dt\bar{\phi}(t) \int_0^T dy\eta(y)\{-\partial_y R(0, t; y, 0)\, \partial_y u_0(y) \\
&\quad + \partial_t R(0, t; y, 0)\, v_0(y)\}, \\
&= \int_0^T dy\eta(y) \left[\partial_y u_0(y) \left(-\partial_y \int_{-\bar{\phi}}^{T} dt\bar{\phi}(t)\, R(0, t; y, 0)\right)\right. \\
&\quad \left. + v_0(y) \left(\int_{-T}^{T} dt\bar{\phi}(t)\, \partial_t R(0, t; y, 0)\right)\right], \\
&= \langle S^\dagger \phi, (u_0, v_0)\rangle_E .
\end{aligned}
$$

To justify the differentiation under the integral sign and integration by parts, in the second and fourth steps, one assumes u smooth, that $\phi = 0$ at $\pm T$, and uses the progressing wave expansion of R—details are left to the reader. The identity holds for general ϕ, (u_0, v_0) by continuity. Note that we have used (4.7) to interchange the spatial variables in R, in the last step. Q.E.D.

LEMMA 2. *Extend the impulse–response trace g to be an odd square-integrable function on $[-T, T]$. Then $g(t) = R(0, t, 0, 0)$ and*

$$[g](0) = \lim_{t \to 0^+} g(t) - \lim_{t \to 0^-} g(t) = 2.$$

If $\eta \in H^1[0, T]$, then g is in H^1_{loc} in any subinterval of $[-T, T]$ not containing zero.

Proof. It is a consequence of the normalization $\eta(0) = 1$ that the impulse–response, as originally defined, under goes a unit jump at $t = 0$. The rest follows from the progressing wave expansion (3.2).

THEOREM 4. *The distribution kernel $G(s, t) = g'(t - s)$ defines a symmetric Hilbert–Schmidt operator of the first kind on $L^2[-T, T]$, and*

$$G = SS^\dagger > 0.$$

Proof. The first statement is an immediate consequence of Lemma 2.

The identity (4.9) reads, for $x = x_0 = 0$ (and using the oddness of R, hence evenness of $D_2 R$, as a function of the second variable),

$$R(0, t-s; 0, 0) = \int dy\eta(y)\{D_2 R(0, s; y, 0)\, R(y, t; 0, 0) - R(0, s, y, 0)\, D_2 R(y, t; 0, 0)\}.$$

On the other hand, according to Lemma 1 and Proposition 2, (4.8), for any $\phi \in L^2[-T, T]$ we have

$$\begin{aligned} SS^\dagger \phi(t) &= \partial_t \left\{ \int_0^T dy\eta(y)\, D_t R(0, t; y, 0) \int_{-T}^T ds\phi(s)\, R(y, s; 0, 0) \right. \\ &\qquad \left. - \int_0^T dy\eta(y)\, R(0, t; y, 0) \int_{-T}^T ds\phi(s)\, D_2 R(y, s; 0, 0) \right\} \\ &= \partial_t \int_{-T}^T ds R(0, t-s; 0, 0)\, \phi(s) \\ &= G\phi(t). \end{aligned}$$

Q.E.D.

COROLLARY 2. $\|S^{-1}\|^{-2} = \inf\{\lambda : \lambda \in \operatorname{Spec} G\}$.

Remark 1. The significance of this result is that the modulus of transparency has been expressed as a property of the impulse–response trace.

Remark 2. Corollary 2 and Theorem 4 imply the necessity of Theorem 1(ii).

Remark 3. These results have been derived under the assumption that η is smooth. They extend *verbatim* to $\eta \in H^1$ by means of arguments similar to those in Section 6 and reference [9], which we omit.

5. A PRIORI ESTIMATES AND PROOFS OF THEOREMS 1 AND 2

We begin by recalling a standard energy identity. Integration of the 1-form

$$\omega = \tfrac{1}{2}[\eta(\partial_t u)^2 + \eta(\partial_x u)^2]\, dx + \eta\partial_x u\partial_t u\, dt$$

around the boundary of the triangle $\{0 \leqslant x \leqslant t \leqslant T\}$ and application of Stokes' theorem leads to the identity

$$\begin{aligned} &\tfrac{1}{2} \int_0^T dx\eta(x)(\partial_t u + \partial_x u)^2(x, x) \\ &\qquad - \tfrac{1}{2} \int_0^T dx\eta(x)[(\partial_t u)^2 + (\partial_x u)^2](x, T) - \int_0^T dt\partial_x u\partial_t u(0, t) \\ &= \iint dx\, dt\, \partial_x u(\eta\partial_t^2 u - \partial_x \eta\partial_x u) \end{aligned}$$

which is justified when $\eta \in H^1[0, T]$. When u is a solution to the wave equation with Neumann condition at $x = 0$, this reduces to

$$\frac{1}{2} \int_0^T dx\, \eta(x) \left(\frac{d}{dx} \bar{u}(x)\right)^2 = \|(u(\cdot, T), \partial_t u(\cdot, T))\|_F^2, \tag{5.1}$$

where $\bar{u}(x) = u(x, x)$. Further, if u is the impulse–response, the first transport equation $\bar{u} = \eta^{-1/2}$ implies

$$\begin{aligned}\frac{1}{2} \int_0^T dx\, \eta \left(\frac{d}{dx} \eta^{-1/2}\right)^2 &= \frac{1}{8} \int_0^T dx \left(\frac{d}{dx} \log \eta\right)^2 \\ &= \|(u(\cdot, T), \partial_t u(\cdot, T))\|_F^2. \end{aligned} \tag{5.2}$$

Finally, the RHS of (5.1) is just the energy of the inverse of the trace operator applied to the impulse–response trace, so we obtain

PROPOSITION 3 (Main a priori estimate). *Suppose that $\eta > 0$, $\eta \in H^1[0, T]$, and let $g \in H^1[0, 2T]$ be the corresponding impulse response and $\varepsilon > 0$ the modulus of transparency (as in Section 4). Then*

$$\int_0^T dx \left(\frac{d}{dx} \log \eta\right)^2 \leqslant 8\varepsilon^{-1} \int_0^{2t} dt \left(\frac{dg}{dt}\right)^2. \tag{5.3}$$

COROLLARY 3. *$\|\eta\|_\infty$, $\|\eta^{-1}\|_\infty$, and $\|\eta\|_{H^1}$ are bounded in terms of $\varepsilon^{-1} \|g\|_{H^1}$.*

What makes this estimate a priori for the inverse problem is, of course, that ε has been identified as a property of the data (Corollary 2).

To prove Theorem 1, we combine estimate (5.3) with the following local existence result:

LEMMA 3. *Suppose $T_1 > T_0$, $f \in H^1[T_0, T_1]$, $g \in L^2[T_0, T_1]$, $\eta(x_0) = f(T_0)^{-2}$, and $0 < \delta < \eta(x_0)$. Then there exists*

$$\Delta = \Delta(T_1 - T_0, \|f\|_{H^1}, \|g\|_{L^2}, \eta(x_0), \delta) > 0$$

and unique functions $\eta \in H^1[x_0, x_0 + \Delta]$, and $u(x, t)$ with

$$\begin{aligned} &\eta(x_0) - \delta < \eta(x) < \eta(x_0) + \delta, \\ &u(x, \cdot) \in H^1[T_0 + x - x_0, T_1 - x + x_0], \\ &\partial_x u(x, \cdot) \in L^2[T_0 + x - x_0, T_1 - x + x_0], \end{aligned}$$

for $x_0 \leqslant x \leqslant x_0 + \Delta$ with the dependence on x in both cases being bounded and measurable, so that

$$\eta \partial_t^2 u - \partial_x \eta \partial_x u = 0, \qquad u(x_0, t) = f(t),$$
$$\partial_x u(x_0, t) = g(t), \qquad u(x, x) = \eta(x)^{-1/2}. \tag{5.4}$$

The proof is given in Section 6.

Now suppose u, η solve boundary value problem (3.4) in the domain $\{(x, t): 0 \leqslant x \leqslant x_0, x \leqslant t \leqslant 2T - x\}$ with $x_0 < T$. It follows from Corollary 1 that u is the impulse response of the portion of the slab lying in $\{0 \leqslant x \leqslant x_0\}$, hence $\{g(t): 0 \leqslant t \leqslant 2x_0\}$ is its trace. The operator G_0 on $L^2[-x_0, x_0]$ with kernel $g'(t-s)$, $|t|$, $|s| \leqslant x_0$, is the restriction of the operator G on $L^2[-T, T]$ in Theorem 1 to $L^2[-x_0, x_0]$, hence is bounded below by the lowest eigenvalue of G:

$$G_0 \geqslant \varepsilon.$$

It follows that the modulus of transparency of the portion of the slab in $\{0 \leqslant x \leqslant x_0\}$ is $\geqslant \varepsilon$ (Corollary 2) and thus by Corollary 3, that η, $1/\eta$, and $\|\eta\|_{H^1}$ are bounded on $[0, x_0]$ by features of the data. It then follows from the main "sideways" energy estimate (Eq. (6.2)) that $\|u(x_0, \cdot)\|_{H^1[x_0, 2T-x_0]}$ and $\|\partial_x u(x_0, \cdot)\|_{L^2[x_0, 2T-x_0]}$ are bounded in terms of the data (T and g), hence independently of x_0. Then Lemma 3 implies that we can choose $\Delta > 0$, independent of x_0, so that the solution u, η of (3.4) can be continued into $[x_0, x_0 + \Delta]$, hence after finitely many repetitions into the entire slab. This completes the proof of Theorem 1.

To prove Theorem 2, we use the local continuity result

LEMMA 4. *Suppose $T_1 > T_0$, $f_1, f_2 \in H^1[T_0, T_1]$, $g_1, g_2 \in L^2[T_0, T_1]$, $0 < \eta_1(x_0) = f_1(T_0)^{-2}$, $\eta_2(x_0) = f_2(T_0)^{-2}$, and $0 < \eta_* < \min\{\eta_1(x_0), \eta_2(x_0)\} \leqslant \max\{\eta_1(x_0), \eta_2(x_0)\} < \eta^*$. Then there exists $\Delta = \Delta(T_1 - T_0, \max(\|f_1\|_{H^1}, \|f_2\|_{H^1}, \|g_1\|_{L^2}, \|g_2\|_{L^2}), \min(\eta_1(x_0), \eta_2(x_0)), \eta_*, \eta^*) > 0$ so that the problems $(i = 1, 2)$*

$$\eta_i \partial_t^2 u_i + \partial_x \eta_i \partial_x u_i = 0 \qquad in \quad \{x_0 \leqslant x \leqslant x_0 + \Delta, T_0 + x \leqslant t \leqslant T_1 - x\},$$
$$u_i(x_0, \cdot) = f_i, \qquad \partial_x u_i(x_0, \cdot) = g_i,$$
$$u_i(x, x) = \eta_i(x)^{-1/2}, \qquad x_0 \leqslant x \leqslant x_0 + \Delta$$

have unique solutions with $\eta_i \in H^1[x_0, x_0 + \Delta]$ $i = 1, 2$, and: $\|\eta_1 - \eta_2\|_{H^1}^2 \leqslant K(\|f_1 - f_2\|_{H^1}^2 + \|g_1 - g_2\|_{L^2}^2)$, where K is a function of the same quantities as Δ.

The proof is given after that of Lemma 3.

The proof of Theorem 2 is now an easy consequence of Lemma 4 and of the main a priori estimate (5.3), which allows uniform estimation of Δ, K independent of x_0, as in the proof of Theorem 1. We leave the details to the reader.

6. Local Solution of the Boundary Value Problem: Proofs of Lemmas 3 and 4

Proof. We wish to solve the problem

(i) $\eta\partial_t^2 u - \partial_x \eta \partial_x u = 0$ in $\{x_0 \leqslant x_1;\ T_0 + x - x_0 \leqslant t \leqslant T_1 - x + x_0\}$,

(ii) $u(x_0, \cdot) = f \in H^1[T_0, T_1]$, $\partial_x u(x_0, \cdot) = g \in L^2[T_0, T_1]$,

(iii) $u(x, T_0 + x - x_0) = \eta^{-1/2}(x)$,

(iv) $u \in L^\infty([x_0, x_1];\ H^1[T_0 + x - x_0, T_1 - x + x_0])$, $\partial_x u \in L^\infty([x_0, x_1]; L^2[T_0 + x - x_0, T_1 - x + x_0])$,

(v) $\eta \in H_1[x_0, x_1]$, $\eta(x_0) = f(T_0)^{-2} > 0$,

(vi) $0 < \eta(x_0) - \delta \leqslant \eta(x) \leqslant \eta(x_0) + \delta$ for $x_0 \leqslant x \leqslant x_1$.

A solution will be found in some interval $[x_0, x_1]$ to the right of x_0. We shall give a lower bound on the length $|x_1 - x_0|$ of this interval in terms of $\|f\|_{H^1}$, $\|g\|_{L^2}$, $\eta(x_0)$, and $\delta > 0$.

To accomplish this, we shall define a mapping from a subset of $H^1[x_0, x_1]$ into $H^1[x_0, x_1]$, so that a solution of (i)–(vi) amounts to a fixed point of the mapping. We shall then show that the map is contractive for $|x_0 - x_1|$ small enough, thus assuring the existence of a fixed point and proving Lemma 3.

Define the set $P_{c,\delta,f} \subset H^1[x_0, x_1]$ by the following conditions:

$$\eta \in P_{c,\delta,f} \quad \textit{iff} \quad \|\eta\|_{H^1} \leqslant C, \quad f(T_0)^{-2} - \delta \leqslant \eta \leqslant f(T_0)^{-2} + \delta.$$

We shall suppose that $C > \|f\|_{H^1}^2 + \|g\|_{L^2}^2$.

For $\eta \in P_{c,\delta,f}$, define $T\eta$ by

$$T\eta(x) = [w(x, T_0 + x - x_0)]^{-2},$$

where w solves the problem

$$\eta\partial_t^2 w - \partial_x \eta \partial_x w = 0$$

in

$$\{x_0 \leqslant x \leqslant x_1,\ T_0 + x - x_0 \leqslant t \leqslant T_1 - x + x_0\},$$

$$w(x_0, \cdot) = f, \qquad \partial_x w(x_0, \cdot) = g. \tag{6.1}$$

The existence of a unique solution w of the problem with $w(x,\cdot)\in H_1$, $\partial_x w(x,\cdot)\in L^2$ may be proven by standard methods, using the "energy" form

$$F_w^2(x)=\tfrac{1}{2}\int_{T_0+x-x_0}^{T_1-x+x_0} dt\,|(\partial_t w)^2+(\partial_x w)^2|(x,t).$$

Thus the wave equation is viewed as a hyperbolic evolution equation in x rather than in t. The main estimate on F for the inhomogeneous problem

$$\eta\partial_t^2 w-\partial_x\eta\partial_x w=h$$

is

$$F(x)\leqslant F(x_0)\exp\left(\frac{(x_1-x_0)^{1/2}}{\eta(x_0)-\delta}\|\eta\|_{H^1}\right)$$
$$+(x_1-x_0)^{1/2}\left(\iint dx\,dt\,h^2\right)^{1/2}. \tag{6.2}$$

The identity whence this estimate is:

$$F^2(x_1)-F^2(x_0)+\frac{1}{2}\int_{x_0}^{x_1} dx(\partial_t w+\partial_x w)^2\,(x,T_0+x-x_0)$$
$$+\frac{1}{2}\int_{x_0}^{x_1} dx(\partial_t w-\partial_x w)^2\,(x,T_1-x+x_0)$$
$$=\frac{1}{2}\int_{x_0}^{x_1} dx(\partial_x\log\eta)\int_{T_0+x-x_0}^{T_1-x+x_0} dt(\partial_x w)^2-\int_{T_0+x-x_0}^{T_1-x+x_0} dt\,\partial_x wh. \tag{6.3}$$

It follows from (6.2) and (6.3) (with $h=0$) that

$$\|\bar{w}\|_D^2\equiv\int_{x_0}^{x_1}(\partial_x\bar{w})^2\leqslant k_1F^2(x_0)\leqslant k_1(\|f\|_{H^1}+\|g\|_{L^2}),$$

where $1\leqslant k_1$ is a smooth increasing function of x_1-x_0, $\|\eta\|_{H^1}$, and $(\eta(x_0)-\delta)^{-1}$, and

$$\bar{w}(x)=w(x,T_0+x-x_0).$$

Thus

$$|\bar{w}(x)-\bar{w}(x_0)|\leqslant k_1(x-x_0)^{1/2}(\|f\|_{H^1}+\|g\|_{L^2}).$$

Now trivial estimates show that $T\eta=\bar{w}^{-2}\in P_{c,\delta,f}$ provided that $|x_1-x_0|$ is smaller than some bound determined by C, δ, $\|f\|_{H^1}$, and $\|g\|_{L^2}$.

To show that T is a contraction mapping, observe that the difference $v = w_1 - w_2$ between solutions of two problems (i)–(iii) with $\eta_1, \eta_2 \in P_{c,\delta,f}$ satisfies

$$\begin{gathered} \partial_t^2 v - \eta_1^{-1} \partial_x \eta_1 \partial_x v = (\partial_x \log \eta_2 - \partial_x \log \eta_1)\, \partial_x w_2, \\ v(x_0, 0) = \partial_x v(x_0, \cdot) = 0. \end{gathered} \tag{6.4}$$

It is easy to show that

$$\|\partial_x \log(\eta_2/\eta_1)\|_{L^2} \leqslant k_2(c, \delta)\, \|\eta_2 - \eta_1\|_{H^1}.$$

Also

$$\int (\partial_x w_2)^2\, dt \leqslant F_{w_2}^2$$

which is in turn bounded by basic estimate (6.2), homogeneous case. So the RHS of (6.4) satisfies

$$\iint dx\, dt \left(\partial_x \log \frac{\eta_2}{\eta_1}\right)^2 w_2^2 \leqslant k_3(c, \delta, f, g) \times \|n_2 - \eta_1\|_{H^1}^2.$$

Now the basic estimate, inhomogeneous case, implies that

$$F_v(x) \leqslant (x_1 - x_0)^{1/2}\, k_4(c, \delta, f, g)\, \|\eta_2 - \eta_1\|_{H^1}, \qquad x_0 \leqslant x \leqslant x_1.$$

This allows one to estimate all of the term in identity (6.3) except the third and fourth, which implies an estimate

$$\begin{aligned} \frac{1}{2} \int_{x_0}^{x_1} \left(\frac{d\bar{v}}{dx}\right)^2 &= \frac{1}{2} \int_{x_0}^{x_1} \left(\frac{d}{dx}\, v(x, T_0 + x - x_0)\right)^2 \\ &\leqslant (x_1 - x_0)\, k_5(c, \delta, f, g)\, \|\eta_2 - \eta_1\|_{H^1}^2. \end{aligned}$$

Since $\bar{v} = \bar{w}_2 - \bar{w}_1$, this estimate implies in turn an estimate for $T\eta_1 - T\eta_2 = \bar{w}_1^{-2} - \bar{w}_2^{-2}$:

$$\|T\eta_1 - T\eta_2\|_{H^1} \leqslant (x_1 - x_0)^{1/2}\, k_6\, \|\eta_1 - \eta_2\|_{H^1}$$

with $k_6 = k_6(c, \delta, f, g)$. Thus for $x_1 - x_0$ sufficiently small, T is a contraction mapping of the bounded set $P_{c,\delta,f} \subset H^1[x_0, x_1]$ into itself, which has a unique fixed point. Since the problem satisfied by a fixed point of T in $P_{c,\delta,f}$ is precisely (i)–(vi), we have proved Lemma 3. Q.E.D.

The proof of Lemma 4 is obtained by an easy application, which we leave to the reader, of the following operator continuity result (notation as in the proof of Lemma 3):

LEMMA 5. *Suppose* $f_i \in H_1[T_0, T_1]$, $g_i \in L_2[T_0, T_1]$, $i = 1, 2$, *with* $f_i(T_0) > 0$. *Then we can select* $x_1 > x_0$ *and a closed bounded set* $P \subset H^1[x_0, x_1]$ *for which* T_{f_i, g_i}, $i = 1, 2$ (*defined as in the proof of Lemma* 4) *map P into itself, are contractive on P, and satisfy*

$$\| T_{f_1, g_1}\eta - T_{f_2 g_2}\eta \|_{H^1}^2 \leqslant k_7(\|f_1 - f_2\|_{H^1}^2 + \|g_1 - g_2\|_{L^2}^2),$$

for $\eta \in P$, *where* k_7 *depends on the same quantities as* K, Δ *in the statement of Lemma* 4.

Proof. We seek P in the form

$$P = \{\eta \in H^1[x_0, x_1] : \|\eta\|_{H^1} \leqslant C, \eta_* \leqslant \eta \leqslant \eta^*\}.$$

For given choices of $C > \max(\|f_i\|_{H^1}^2 + \|g_i\|_{L^2}^2)\ i = 1, 2, 0 < \eta_* < \min_{i=1,2}(f_i(T_0)) \leqslant \max_{i=1,2} f_i(T_0) < \eta^*$, we can select $x_1 > x_0$ as in the proof of Lemma 3 so that T_{f_1, g_1}, T_{f_2, g_2} map P into itself and are contractive.

For $\eta \in P$ denote by $v = v[\eta]$ the solution of

$$\eta \partial_t v - \partial_x \eta \partial_x v = 0, \qquad v(x_0, \cdot) = f_1 - f_2, \qquad \partial_x v(x_0, \cdot) = g_1 - g_2,$$

in the obvious domain. Then the "sideways" energy estimate (6.2) implies $F_v(x) \leqslant k_8(\|f_1 - f_2\|_{H^1}^2 + \|g_1 - g_2\|_{L^2}^2)$ for $x_0 \leqslant x \leqslant x_1$, where k_8 is estimated in terms of $x_1 - x_0$, C, η_*^{-}, η^*. Identity (6.3) then implies that

$$\int_{x_0}^{x_1} dx (\bar{v}')^2 \leqslant k(\|f_1 - f_2\|_{H^1}^2 + \|g_1 - g_2\|_{L^2}^2),$$

where $\bar{v}(x) = v(x, T_0 + x - x_0)$. Since $\bar{v} = (T_{f_1, g_1}\eta)^{-1/2} - (T_{f_2, g_2}\eta)^{-1/2}$, an estimate of the stated form follows immediately. Q.E.D.

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